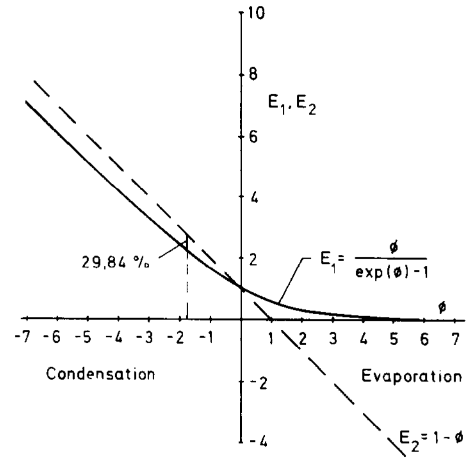

FIG. 2. The variation of the enhancement factors for the interfacial shear stress E_1 and E_2 with the mass transfer rate.

Both the equations possess the equal limits for $\Phi = 0$ and $\Phi \to -\infty$, but equations (3) and (4) for $\Phi \to +\infty$ yield $E_1 = 0$ whereas equation (6) yields $E_2 \to -\infty$. This $(-\infty)$, a change in sign, is not at all understandable. That means with conditions $\Phi > 0$ (by blowing velocity of gas phase) equation (6) produces the reversion of the shear stress, which is not at all feasible.

A negative abscissa of Φ, the uncoupled solution [equation (6)], predicts a greater enhancement factor with a maximum of 30% at $\Phi = -1.8$ as compared with the coupled solution [equation (3)]. In Fig. 2 the course of both equations can be seen. For practical purposes and in the case where $\bar{U}$ is much greater when compared to u_o, u_o can be neglected but not otherwise. It is recommended when the velocity of the steam phase when compared to the velocity of the interphase liquid–gas is much greater and not for small differences, i.e. only when $u_o/\bar{U} \to 0$.

For the case that $(\bar{U} - u_o)$ goes to zero E_1 and E_2 tends to infinity but $\tau_o^{\bullet}$ in both cases tends to zero as can be seen immediately from equations (3)–(6).

REFERENCES

1. J. H. Linehan, M. Petrick and M. M. El-Wakil, On the interface shear stress in annular flow condensation, *J. Heat Transfer* **91C**(3), 450–452 (1969).
2. Y. R. Mayhew and J. K. Aggarwal, Laminar film condensation with vapour-drag on a flat surface, *Int. J. Heat Mass Transfer* **16**, 1944–1949 (1973).
3. M. Soliman, J. R. Schuster and P. S. Berenson, A general heat transfer correlation for annular flow condensation, *J. Heat Transfer* **90C**, 267–276 (1968).
4. Discussion to W. M. Rohsenow, J. H. Webber and A. T. Ling, Effect of vapor velocity on laminar and turbulent film condensation, *Trans. Am. Soc. Mech. Engrs* **78**, 1637–1643 (1956).
5. M. D. Razavi and A. S. Damle, Heat transfer coefficients for turbulent filmwise condensation, *Trans. Instn Chem. Engrs* **56**, 81–85 (1978).
6. R. B. Bird, W. E. Stewart and E. N. Lightfoot, *Transport Phenomena*, Chapter 21, pp. 642–666. Wiley, New York (1960).
7. G. Ackermann, Wärmeübergang und molekulare Stoffübertragung im gleichen Feld bei großen Partialdruckdifferenzen, *ForschHft Ver. Dt. Ingr.* 382 (1937).

Int. J. Heat Mass Transfer. Vol. 23, pp. 1695–1698
0017-9310/80/1201-1695 $02.00/0

TRANSIENT RESPONSE OF FINS BY COORDINATE PERTURBATION EXPANSION

A. AZIZ

Department of Mechanical Engineering, College of Engineering, P.O. Box 800, Riyadh, Saudi Arabia

and

T. Y. NA

Department of Mechanical Engineering, University of Michigan-Dearborn, Dearborn, Michigan 48128, U.S.A.

(*Received* 18 *March* 1979 *and in revised form* 24 *June* 1980)

NOMENCLATURE

- b, fin thickness;
- c, specific heat;
- E, emissivity;
- h, heat-transfer coefficient;
- k, thermal conductivity;
- m, exponent of power law;
- N, fin parameter;
- q, heat-transfer rate;
- Q, dimensionless heat-transfer rate, $qx_0/bk\ (T_b - T_e)$;
- t, time;
- T, temperature;
- T_b, fin base temperature;
- T_e, environment temperature;
- x, distance from fin base;
- x_0, reference length;
- X, dimensionless distance, x/x_0.

Greek symbols

- α, thermal diffusivity, $k/\rho c$;

η, similarity variable, $X/2\sqrt{\tau}$;
θ, dimensionless temperature;
σ, Stefan–Boltzmann constant;
ρ, density;
τ, dimensionless time, $\alpha t/x_0^2$;
ε, perturbation parameter, $4N^2\tau$.

INTRODUCTION

THE TRANSIENT behaviour of fins is of importance in applications such as electronic components, solar collectors, radiators on space vehicles. One basic problem is to predict how the fin responds when its base temperature undergoes a step change. For convecting fins with linear cooling law, the method of separation of variables can be applied to obtain the solution in series form. Such solutions have been reported by Donaldson and Shouman [1] for a straight fin and by Chapman [2] for an annular fin. The drawback with these solutions is that for small values of time the convergence is rather slow. Therefore as an alternative, Suryanarayana [3] used the Laplace transform technique which enabled him to derive rapidly convergent approximate solution for the early part of the transient.

In some fin applications the analysis based on linear cooling law is not applicable because the process is governed by a power law type dependence on temperature, i.e. θ^m. For example, the exponent m takes the values of 0.75, 1.25, 3.0 and 4.0 when the fin is cooled due to film boiling, natural convection, nucleate boiling and radiation to space at absolute zero, respectively [4]. Of these, only the case of radiating fin has been treated in the literature. The papers by Okamoto and Negishi [5] and Russell and Chapman [6] may be cited as representative contributions. In [5] a numerical scheme is used to predict the transient response of a finned tube-sheet radiator while in [6] the method of free-parameter is used to determine the base temperature–time variations which permit similarity solutions for an infinitely long radiating fin.

With the foregoing in perspective, it seems desirable to develop a method of analysis for the transient response of a fin with a power law type cooling process. Such an analysis is presented in this paper using a coordinate perturbation expansion in dimensionless time. For the case of convecting fin ($m = 1$), exact solutions of the sequence of perturbation equations are given in terms of repeated integrals of error function and the perturbation solution is shown to be identical to the exact solution derived using Laplace transform. For other values of m, analytical solutions of the perturbation equations do not seem feasible. These are solved numerically using the method of superposition [7] and results for $m = 2, 3$ and 4 are tabulated. In each case the perturbation expansion is terminated at the third term and its range of applicability is subsequently increased with Shanks transformation. From the temperature solutions, series for base heat flux are derived. It will be seen that the analysis gives accurate predictions over the major portion of the transient period.

ANALYSIS

Formulation

Consider one-dimensional conduction in a semi-infinite fin of uniform thickness, b. Let the fin be initially at the environment temperature T_e. At time $t = 0$, the base temperature is suddenly changed to T_b. We assume that the heat dissipated from the fin surfaces follows a power law type dependence on temperature difference. As remarked earlier, this makes the analysis applicable to a number of physical situations. The choice of a semi-infinite geometry permits the transformation of the governing nonlinear partial differential equations into a sequence of similarity type linear perturbation equations. The applicability of the results to finite fins will be discussed later. For constant thermal properties, the governing equation for the transient response can be written in dimensionless form as

$$\frac{\partial^2\theta}{\partial X^2} - N^2\theta^m = \frac{\partial\theta}{\partial\tau} \tag{1}$$

$$\theta(0,\tau) = 1, \quad \theta(X,0) = \theta(\infty,\tau) = 0 \tag{2}$$

where except for N and θ the symbols are as defined in the nomenclature. The parameter N and θ are defined appropriately in accordance with the mechanism of surface heat transfer. For example, with convecting fin ($m = 1$), $N^2 = 2hx_0^2/bk$, $\theta = (T - T_e)/(T_b - T_e)$ and for a fin radiating to zero environment temperature ($m = 4$), $N^2 = 2E\sigma T_b^3x_0^2/bk$ and $\theta = T/T_b$.

Coordinate perturbation solution

Let us assume a perturbation expansion in dimensionless time ε as

$$\theta = \sum_{n=0}^{\infty} \varepsilon^n\theta_n(\eta) \tag{3}$$

where $\eta = X/2\sqrt{\tau}$ and $\varepsilon = 4N^2\tau$. Substituting equation (3) into equations (1) and (2) gives

$$\sum_{n=0}^{\infty} \varepsilon^n(\theta_n'' + 2\eta\theta_n' - 4n\theta_n) = \varepsilon\left(\sum_{n=0}^{\infty} \varepsilon^n\theta_n\right)^m \tag{4}$$

$$\eta = 0, \quad \theta_0 = 1, \quad \theta_n = 0, \quad n = 1,2,3,\ldots \tag{5}$$

$$\eta = \infty, \quad \theta_n = 0, \quad n = 0,1,2,\ldots \tag{6}$$

In equation (4) primes denote differentiation with respect to η. We shall carry out the expansion to three terms. Equating coefficients of ε^0, ε^1 and ε^2, we have

$$\varepsilon^0: \ \theta_0'' + 2\eta\theta_0' = 0 \tag{7}$$

$$\eta = 0, \quad \theta_0 = 1; \quad \eta = \infty, \quad \theta_0 = 0 \tag{8}$$

$$\varepsilon^1: \ \theta_1'' + 2\eta\theta_1' - 4\theta_1 = \theta_0^m \tag{9}$$

$$\eta = 0, \quad \theta_1 = 0; \quad \eta = \infty, \quad \theta_1 = 0 \tag{10}$$

$$\varepsilon^2: \ \theta_2'' + 2\eta\theta_2' - 8\theta_2 = m\theta_0^{m-1}\theta_1 \tag{11}$$

$$\eta = 0, \quad \theta_2 = 0; \quad \eta = \infty, \quad \theta_2 = 0. \tag{12}$$

The zero-order problem defined by equations (7) and (8) is recognised as the classical problem of transient conduction into a semi-infinite solid. For the solutions of equations (9)–(12) we first concentrate on the case of convecting fin ($m = 1$) and demonstrate that the perturbation solution is identical to the exact solution which can be obtained with the aid of Laplace transform. For $m = 1$, equations (9) and (10) and (11) and (12) can be solved in terms of repeated integrals of error function. Omitting the intermediate details, we give the 3 term perturbation solution as

$$\theta = \operatorname{erfc}\eta + \varepsilon\left(i^2\operatorname{erfc}\eta - \frac{1}{4}\operatorname{erfc}\eta\right) + \varepsilon^2\left(i^4\operatorname{erfc}\eta - \frac{1}{4}i^2\operatorname{erfc}\eta + \frac{1}{32}\operatorname{erfc}\eta\right) \tag{13}$$

where $i^n\operatorname{erfc}\eta$ is the nth repeated integral of error function.

The base heat-transfer rate in dimensionless form is given by

$$Q = \frac{qx_0}{bk(T_b - T_e)} = -\frac{1}{2\sqrt{\tau}}\theta'(0). \tag{14}$$

Using equation (13) to evaluate $\theta'(0)$ we have

$$Q\sqrt{\pi\tau} = 1 + \frac{1}{4}\varepsilon - \frac{1}{96}\varepsilon^2. \tag{15}$$

The solution of equations (1) and (2) for $m = 1$ can be obtained with the aid of Laplace transform and appears in terms of ε and η as

$$\theta = \frac{1}{2}\left[e^{-\eta\varepsilon^{1/2}}\operatorname{erfc}\left(\eta - \frac{1}{2}\varepsilon^{1/2}\right) + e^{\eta\varepsilon^{1/2}}\operatorname{erfc}\left(\eta + \frac{1}{2}\varepsilon^{1/2}\right)\right] \quad (16)$$

from which the heat-transfer rate follows as

$$Q\sqrt{\pi\tau} = e^{-(1/4)\varepsilon} + \frac{\sqrt{\pi}}{2}\varepsilon^{1/2}\operatorname{erfc}\left(\frac{1}{2}\varepsilon^{1/2}\right). \quad (17)$$

Using the series expansion expansion for the error function appearing in equation (17), it is easy to demonstrate that the perturbation solution (15) is identical to the exact solution (17). Although we have given only three terms in equation (15), we calculated two additional terms and found them to be identical to the corresponding terms of expanded version of equation (17).

Deferring the discussion of the accuracy of the truncated perturbation expansion, we now consider the solutions of equations (9) and (10) and (11 and 12) for $m = 2, 3$ and 4. It did not appear feasible to attempt exact analytical solution of these equations and were therefore solved numerically using the method of superposition detailed out in [7]. The solutions are briefly tabulated in Table 1. From the numerical solutions the heat-transfer rate series follow as

$$m = 2: \; Q\sqrt{\pi\tau} = 1 + 0.183617\varepsilon - 0.010246\varepsilon^2 \quad (18)$$

$$m = 3: \; Q\sqrt{\pi\tau} = 1 + 0.147253\varepsilon - 0.010076\varepsilon^2 \quad (19)$$

$$m = 4: \; Q\sqrt{\pi\tau} = 1 + 0.123757\varepsilon - 0.007081\varepsilon^2. \quad (20)$$

DISCUSSION

For brevity, we concentrate our discussion on the heat-transfer rate, which is of main interest, rather than the temperature distribution. Examining the case of convecting fin, equation (15), it is seen that the coefficients of the terms diminish very rapidly and therefore the solution may be expected to be valid even for $\varepsilon > 1$ although ε is originally stipulated to be less than unity. A simple calculation shows that up to $\varepsilon = 4$, equation (15) gives negligible error compared to the exact solution. However, making a still bolder application and using $\varepsilon = 10$, the error becomes 13% and thus the usefulness of the solution is limited. To extend its range of applicability we apply Shanks transformation [8] to equation (15) to obtain

$$Q\sqrt{\pi\tau} = \frac{1 + 0.291667\varepsilon}{1 + 0.041670\varepsilon}. \quad (21)$$

Equation (21) agrees to within 3% with the exact solution up to $\varepsilon = 10$ and within 10% up to $\varepsilon = 30$ which obviously shows a significant improvement over the original solution. To assess the accuracy of equation (21) we choose $N = 5$ keeping in mind that for $N \geq 3$ a finite fin rapidly approaches the behaviour of an infinitely long fin. A plot of Q vs τ based on equation (21) is compared with the exact solution in Fig. 1 where the steady-state value Q_{ss} is also indicated. It is evident that equation (21) gives a fairly accurate prediction over the entire transient period. However, it should be noted that the truncated expansion represented by equation (15) or (21) does not approach the steady-state value as equation (17) does for $\varepsilon \to \infty$.

The success of Shanks transformation in the foregoing case encourages us to apply it to equations (18)–(20). The results follow as

$$m = 2: \quad Q\sqrt{\pi\tau} = \frac{1 + 0.239418\varepsilon}{1 + 0.055801\varepsilon} \quad (22)$$

$$m = 3: \quad Q\sqrt{\pi\tau} = \frac{1 + 0.215679\varepsilon}{1 + 0.068426\varepsilon} \quad (23)$$

$$m = 4: \; Q\sqrt{\pi\tau} = \frac{1 + 0.180976\varepsilon}{1 + 0.057216\varepsilon}. \quad (24)$$

In order to assess the accuracy of equations (22)–(24) a numerical solution of equations (1) and (2) had to be carried out on HP 3000 using an implicit finite difference scheme. These results are shown in Fig. 1 together with the results from equations (22)–(24). Also indicated are the steady-state values. Again, though equations (22)–(24) do not approach Q_{ss} as $\varepsilon \to \infty$, they cover the major portion of the transient accurately. In general, these equations are quite accurate up to the time when the transient heat flux is within about 7% of the steady-state value.

The response curves of Fig. 1 clearly show the influence of the exponent m. During the early part of the transient, m does not have much effect. This is understandable physically because initially the energy leaving the fin base contributes mainly to increase the internal energy of the fin, and the surface transfer mechanism exerts little influence. However, in the latter part of the transient the influence of m becomes quite significant.

Regarding the assumption of infinite length, the analysis is quite appropriate for a finite fin when it is heated by laminar condensation or cooled by nucleate boiling because then a good fin design requires $N \geq 4$ as shown in [9, 10]. In any case, for N beyond 3 a finite fin rapidly approaches the behaviour of infinitely long fin, the results for N around 3 would not be grossly in error and should be useful at least for preliminary calculations. In situations, where the values of N encountered do not justify the assumption, the present

Table 1. Solutions for θ_1 and θ_2

	$m = 2$		$m = 3$		$m = 4$	
η	$\theta_1 \times 10$	$\theta_2 \times 10^2$	$\theta_1 \times 10$	$\theta_2 \times 10^2$	$\theta_1 \times 10$	$\theta_2 \times 10^2$
0	0	0	0	0	0	0
0.2	−0.273010	0.315857	−0.194087	0.185854	−0.145910	0.120772
0.4	−0.338522	0.459346	−0.215816	0.236874	−0.148677	0.137742
0.6	−0.296490	0.440739	−0.172802	0.200760	−0.112428	0.107797
0.8	−0.217648	0.340797	−0.118676	0.140994	−0.074876	0.072459
1.0	−0.141389	0.227849	−0.073745	0.088315	−0.045903	0.044515
1.4	−0.045014	0.074218	−0.022532	0.027072	−0.013935	0.013520
1.8	−0.010389	0.017218	−0.005146	0.006184	−0.003181	0.003087
2.0	−0.004448	0.007378	−0.002201	0.002646	−0.001360	0.001320
2.5	−0.000368	0.000611	−0.000182	0.000219	−0.000112	0.000109
3.0	−0.000000	0.000000	−0.000000	0.000000	−0.000000	0.000000

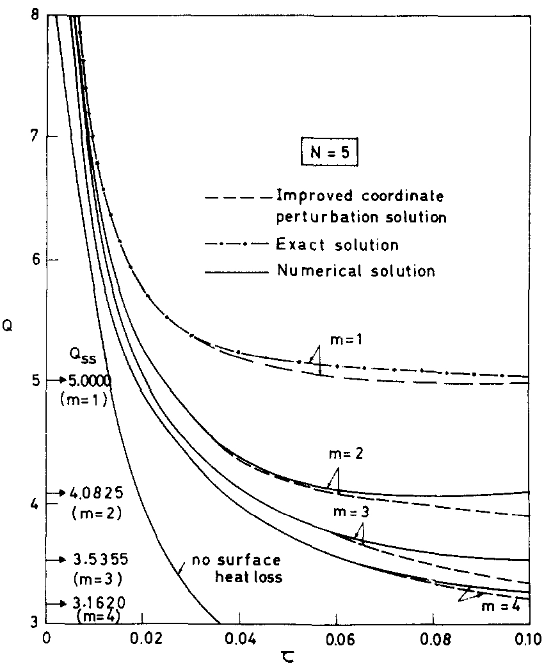

FIG. 1. Transient heat transfer rate.

pattern of results can be used judiciously with results for finite convecting fins (available in [3] for N in the range 0.01 to 10) to deduce the corresponding results for the nonlinear cases. We have carried out this exercise and found that such estimates are fairly reasonable if not very accurate.

REFERENCES

1. A. B. Donaldson and A. R. Shouman, Unsteady-state temperature distribution in a convecting fin of constant area, *Appl. Scient. Res.* **26,** 75–85 (1972).
2. A. J. Chapman, Transient heat conduction in annular fins of uniform thickness, *Chem. Engng Symp. Ser.* **55,** 195–201 (1959).
3. N. V. Suryanarayana, Transient response of fins, *J. Heat Transfer* **97C,** 417–423 (1975).
4. K. W. Haley and J. W. Westward, Boiling heat transfer from single fins, *Proc. 3rd Int. Heat Transfer Conference*, Chicago, Vol. 3, pp. 245–253 (1966).
5. Y. Okamoto and K. Negishi, Transient response of finned tube-sheet radiator for space power plants, *Trans. Japan Soc. Aero. Space Sci.* **12,** 73–79 (1969).
6. L. D. Russell and A. J. Chapman, Analytical solution for transient flow of energy in one-dimensional radiating fin, *AIAA Jl* **6,** 90–93 (1968).
7. T. Y. Na, *Computational Methods in Engineering Boundary Value Problems*, Chapter 2. Academic Press, New York (1979).
8. M. Van Dyke, *Perturbation Methods in Fluid Mechanics*, Annotated edition, p. 205. Parabolic Press, Stanford (1975).
9. J. H. Lienhard and V. K. Dhir, Laminar film condensation on nonisothermal and arbitrary-heat flux surfaces, and on fins, *J. Heat Transfer* **96C,** 197–203 (1974).
10. A. Aziz and J. Y. Benzies, Application of perturbation techniques to heat transfer problems with variable thermal properties, *Int. J. Heat Mass Transfer* **19,** 271–276 (1976).